Consistency analysis of Clough-Tocher macro-elements

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Abstract

In this paper, we are concerned with Clough-Tocher macro-elements of any smoothness in any dimension. An *n* dimensional Clough-Tocher complex is a split of an *n* simplex by connecting its centroid with its vertices. By using the Bernstein-Bézier representation of polynomials, we first make a unified analysis of the data compatibility in bivariate macro-element and a rule is given in a general case that the underlying macro-triangle can be wildly subdivided. Then Clough-Tocher macro-elements are setup in a recursive way that the *n* dimensional case can be obtained based on the $n-1$ dimensional case with $n \geq 3$.

Keywords: Bernstein-Bezier form, interpolation, multivariate spines, macro-elements

1 Introduction

In this paper, we are concerned with local interpolation over triangulations. By local we mean that the piece of the interpolant over a triangle depends only on data assigned on that triangle. Such an interpolant has received widespread discussion as a tool for use in free-form surface design in the Computer Aided Geometric Design, finite element computation, and scattered data processing.

The general setting for local interpolation of a prescribed smoothness over triangulations is to set up a model on one single representative triangle. It has a triangular symmetry in the sense that on the model triangle, the data at the vertices (or on its edges) have the same types and rotating the triangle with respect to its vertices does not alter the properties of the interpolant. In order to attain global smoothness on the whole triangulation, a successful model must allow the interpolants on any two adjacent triangles satisfying the prescribed smoothness across the common edge. For the sake of computation, the interpolant has to be simple enough and a good candidate of it can be a polynomial, a rational polynomial, and even their piecewise versions defined on a split of the model triangle.

In literature, there have been works on lower dimensional cases; see [1] and [2]. Originally, Courant was the first to construct the triangular C^0 polynomial element, followed by Argiris for the C^1 element, by Wehlan[3] for the C^2 element, and by Zenisek [4] for the general C^r polynomial elements. The polynomial interpolant is simple but has the drawback that its degree has to be as larger as 4 times of the smoothness. To deal with this, a smart trick is to split the triangle into pieces and to take piecewise polynomials as the interpolant, which degree could be significantly lowered. This has triggered intensive research on 2D case; see [5] for a concise reference. Nonetheless, there are few such interpolation schemes for higher dimensions.

A key obstacle in constructing triangular elements is the consistency of the smoothness, interpolation conditions and the degrees of piecewise polynomials. The interpolation condition means the data assigned at the vertices and on the edges, which are derivatives in general. In this paper, we will focus on interpolation on Clough-Tocher macro-elements in higher dimensions. A Clough-Tocher complex in R^n is a split of an *n*-simplex by connecting its centroid with its vertices. We will setup the relationship amongst the amoothness, the order of derivatives and the degree of polynomials.

This paper is organized as follows. In section 2, preliminary concepts are presented. Compatibility of Hermite interpolations with polynomials and piecewise polynomials are discussed respectively in sections 3 and 4, while the analysis of Clough-Tocher elements is made in section 5. We conclude this work in section 6.

2 Preliminaries

2.1 NOTATIONS

Throughout this paper, we use the standard notation R^n to denote the Euclidean space of dimension *n* and and P_k for the space of polynomials of degree *k*. Still, the set Z_+^n of *n*-tuple non-negative integers will be utilized as multi-indices. For any point $x=(x_1,x_2,\cdots,x_n)$ in R^n or in Z_+^n, the weight of *x* is meant as $|x|=\sum_{i=1}^n x_i$.

Denote by $I_n=\{0,\ 1,\ \cdots,\ n\}$ for an integer $n \geq 0$. Suppose $A_i \in R^n$, $i \in I_n$, are points in general position. If

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$m \le n$ and $J = \{j_0, j_1, \cdots, j_m\} \subset I_n$, denote by $\sigma_J = \sigma_{j_0, j_1, \cdots, j_m}$ the simplex $A_{j_0} A_{j_1} \cdots A_{j_m}$, which is an m-face of the n-simplex $\sigma = \sigma_{I_n}$. Still, if $K = J^c = I \setminus J$, denote by $\hat{\sigma}_J = \sigma_K$ the opposite face of σ_J. Denote $\Lambda_k = \{\lambda \in Z_+^{n+1} : |\lambda| = k\}$. Given $J = \{j_0, j_1, \cdots, j_m\} \subset I_n$, denote by

$$R_k^r(J) = \{\lambda \in \Lambda_k : \ k - r \le \sum_{i=0}^{m} \lambda_{j_i} \le k\}$$

the r influence region of σ_J, and by $L_k^r(J)$ the r influence layer of σ_J defined as

$$L_k^0(J) = R_k^0(J),$$
$$L_k^r(J) = R_k^r(J) \setminus R_k^{r-1}(J), \quad \text{for} \quad r > 0.$$

Further, let

$$\tilde{L}_k^{r_0, r_1, \cdots, r_m}(J) = L_k^{r_m}(J) \setminus \bigcup_{l=0}^{m} \bigcup_{K \subset J, |K| = l+1} L_K^{r_l}(J)$$
$$\tilde{R}_k^{r_0, r_1, \cdots, r_m}(J) = R_k^{r_m}(J) \setminus \bigcup_{l=0}^{m} \bigcup_{K \subset J, |K| = l+1} R_K^{r_l}(J).$$

Then it's easy to obtain the following result.

Proposition 2.1

If $k \ge 2r_0 + 1$, *and* $r_l \ge 2r_l + 1$, $l = 0, 1, \ldots, j_{m-1}$, *then the collection*

$$\{\ \tilde{R}_k^{r_0, r_1, \cdots, r_i}(J) \ : \ J \subset I_n, |J| = i + 1\ \}$$

forms a partition of Λ_k. *Precisely,*

$$\tilde{R}_k^{r_0, r_1, \cdots, r_m}(i_0, i_1, \cdots, i_m) \cap \tilde{R}_k^{r_0, r_1, \cdots, r_m}(j_0, j_1, \cdots, j_m) = \varnothing$$

for $\{i_0, i_1, \cdots, i_m\} \ne \{j_0, j_1, \cdots, j_m\}$, *if*

$$r_l \ge 2r_{l+1}, \ l = 0, 1, \ \cdots, \ j_{m-1}, \ \text{ and } k \ge 2r + 1.$$

2.2 BEZIER POLYNOMIALS

It's well-known [5] that any point $A \in R^n$, with Cartesian coordinates $x = (x_1, x_2, \ldots, x_n)$, has an alternative representtation in barycentric coordinates $u = (u_0, u_1, \ldots, u_n)$ with respect to the simplex σ_{I_n} in such a way that

$$A = \sum_{i \in I_n} u_i A_i, \quad \text{with} \quad \sum_{i \in I_n} u_i = 1. \tag{2.1}$$

Due to this, we use both u and x to indicate the point A and, given a function f, $f(u) = f(x)$ for the evaluation of f at A; this won't be confusing since it's clear in its context.

Given a multi-index $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n) \in Z_+^{n+1}$ with weight $|\lambda| = \sum_{i \in I_n} \lambda_i$, define the Bernstein polynomial with λ

$$B_\lambda(u) = \binom{|\lambda|}{\lambda} u^\lambda,$$

where

$$\binom{|\lambda|}{\lambda} = \frac{\lambda!}{\lambda_0! \lambda_1! \cdots \lambda_n!}, \quad u^\lambda = \prod_{i \in I_n} u_i^{\lambda_i}.$$

Denote by

$$\sigma(\lambda) = \sum_{i \in I_n} \lambda_i A_i / n$$

the domain point in σ with index λ.

If $J = \{j_0, j_1 \ldots j_m\} \subset I$ and the index λ is such that its components with indices J^c are zero, then it will be convenient to take $\sigma_J(\lambda^m) = \sigma(\lambda)$, where

$$\lambda^m = (\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_m}) \in Z_+^m.$$

Let $P_k = P_k(R^n)$ denote the polynomials in n-variables of degrees up to k. With the barycentric coordinates system on the simplex σ, a polynomial $p \in P_k$ can be written in the well-known Bernstein-Bezier form or B-form for short:

$$p(u) = \sum_{|\lambda| = k} b_\lambda B_\lambda(u),$$

where b_λ is the Bezier-ordinate with the domain point $\sigma(\lambda)$.

Let $B \in R^n$ be another point with cartesian coordinates y and barycentric coordinates v. Then the vector AB is expressed as $y - x$ and $v - u$. Notice that $v - u$ has weight 0 and therefore an element in R^{n+1} with weight 0 is regarded as a vector in R^n that is expressed in barycentric coordinates and the evaluation of B_λ at such a vector does formally make sense.

Now let $\delta = (\delta_0, \delta_1, \ldots, \delta_n)$ be a vector in R^n. If $f \in C^r$, let $\partial_\delta^r f$ denote the r-th derivative of f along the direction δ, which is meaningful since it's not difficult to derive that

$$\partial_\alpha f(u) = \sum_{i \in I_n} \delta_i \frac{\partial f(u)}{\partial u_i} = \sum_{i=1}^{n} v_i \frac{\partial f(x)}{\partial x_i},$$

where $(v_1, v_2, \ldots, v_n)$ stand for the cartesian coordinates of the vector δ.

One great benefit we can get from the Bezier representation is that the de Casteljau algorithm holds

$$\begin{gathered} b_\lambda^0(u) = b_\lambda, \ \ b_\lambda^{r+1}(u) = \sum_{|\varepsilon| = 1} B_\varepsilon^1(u) b_{\lambda+\varepsilon}^r(u), \\ \lambda, \varepsilon \in Z_+^{n+1}, |\lambda| \ = n - r, \ \ r = 0, \ 1, \ \cdots, \ k, \end{gathered} \tag{2.2}$$

which provides a stable way calculating the Bezier polynomial. In fact, we have $p(u) = b_\lambda^k(u)$, $|\lambda| = 0$. Still, de Casteljau algorithm helps in finding derivatives of a polynomial. Let δ be a vector in R^n in barycentric coordinates. Then

$$\partial_\delta^r(u) = \frac{k!}{(k-r)!} \sum_{|\lambda| = r} b_\lambda^{k-r}(u) B_\lambda^r(\delta).$$

With this formulation, one can analyse the smooth relation between two polynomial patches defines on two adjacent simplices [6].

From the discussion above, we can also see that the compatibility holds for $r_i = 2r_{i+1}$, $l = 0, 1,\ldots, j_{n-2}$, and $r_{n-1} = \mu$. Therefore we conclude that

Theorem 2.2

The C^μ polynomial element on an n simplex σ can be realized by a polynomial of degree $k = 2^{n-1}\mu + 1$ and the data assigned on any r-face of σ are of order $2^{n-1-r}\mu$. And these choices are optimal.

3 Compatibility of Hermite interpolations with polynomials

3.1 THE ONE-DIMENSIONAL CASE

In the one-dimensional case, recall the following Hermite interpolation problem:

For $f \in C[a, b]$, find polynomial $p \in P_k$ such that

$$\frac{d^t}{dx^t}p(x) = \frac{d^t}{dx^t}f(x), \quad \text{for } x = a, b, \quad 0 \le t \le r. \qquad (3.1)$$

It's known in [6] that the data assigned at the endpoints a and b are consistent if and only if the number of conditions does not exceed the dimension of P_k, i.e.,

$$k + 1 \ge 2(r + 1) \text{ or } k \ge 2r + 1.$$

If k is taken to be $2r + 1$, the solution to the above interpolation (3.1) is unique.

3.2 THE n-DIMENSIONAL CASE

Let $\sigma = A_0A_1\cdots A_n$ be an n simplex with vertices $A_1, A_2,\ldots, A_n$ in general positions. For $0 \le i_0 < i_1 < \cdots < i_m$, denote by $\sigma_{i_0,i_1,\cdots,i_m} = A_{i_0}A_{i_1}\cdots A_{i_m}$ the m-face of σ with indices $i_0, i_1, \ldots, i_m$, by

$$R^r_{i_0,i_1,\cdots,i_m} = \{\sum_{i=0}^{n}\lambda_j A_i / n : \sum_{i=0}^{n}\lambda_i = n, \quad n - r \le \sum_{j=0}^{m}\lambda_{i_j} \le n\}$$

the r influence region of $\sigma_{i_0,i_1,\cdots,i_m}$ and by $L^r_{i_0,i_1,\cdots,i_m}$ the r influence layer of $\sigma_{i_0,i_1,\cdots,i_m}$, defined as

$$L^0_{i_0,i_1,\cdots,i_m} = R^0_{i_0,i_1,\cdots,i_m},$$
$$L^r_{i_0,i_1,\cdots,i_m} = R^r_{i_0,i_1,\cdots,i_m} \setminus R^r_{i_0,i_1,\cdots,i_m}, \quad for \quad r > 0.$$

Suppose $p \in P_k(R^n)$ and r_i, $i = 0, 1, \cdots, n-1$ are non-negative integers. Assign on each i-face of σ data of order r_i as follows:

1) r_0 data at A_j with influence region R_j;
2) r_1 data on A_jA_k with influence region

$$R^0_{j,k} = R^0_j \setminus (R_j \bigcup R_k);$$

3) r_i data on any i-face $\sigma_{j_1,j_2,\cdots j_i} = A_{j_1}A_{j_2}\cdots A_{j_i}$ of σ with influence region

$$R^0_{j_1,j_2,\cdots,j_i} = R_{j_1,j_2,\cdots,j_i} \setminus \bigcup_{\{l_1,l_2,\cdots,l_{i-1}\}\subset\{j_1,j_2,\cdots,j_i\}} R_{l_1,l_2,\cdots,l_{i-1}}.$$

3.3 COMPATIBILITY OF POLYNOMIAL ELEMENTS

In this part, we are discussing the compatibility of data given on an n-simplex $\sigma_n = A_0A_1\cdots A_n$.

We start from the case $n = 2$. Let $T = A_0A_1A_2$ be a triangle. A polynomial $p \in P_m$ on T satisfies

1) vertex data of order r;
2) edge data of order μ.

Suppose p at A_0 has B-ordinates b_λ, $|\lambda| = m$. The compatibility means that the influenced B-ordinates on the edges E_1 and E_2 are compatible with those at A_0 so that $\Lambda = \{\lambda : \lambda_1 \le \mu\} \bigcup \{\lambda : \lambda_1 \le \mu\}$ are dependent on $\{\lambda : \lambda_0 \ge m - r\}$. Since $(m - 2r, r, r) \in \Lambda$ and $2r$ is the farthest layer that belongs to Λ, we have [4]

Theorem 3.1 *It holds for 2-simplex that $r \ge 2\mu$ and $m \ge 4\mu + 1$. There exist schemes with $r = 2\mu$ and $m = 4\mu + 1$.*

Now suppose on an n-simplex, there exist a scheme with $r_n = 2^{n-1}\mu$ and $m_n = 2^n\mu + 1$. We will show that on an $(n + 1)$-simplex there exist a scheme with $r_{n+1} = 2^n\mu$ and $m_{n+1} = 2^{n+1}\mu + 1$.

Consider the $r + 1$ layer of p at A_0, which can be regarded as a polynomial of degree $r + 1$ defined on an n-simplex. The vertex data are of order r_n and the polynomial is of degree m_n. Therefore we have $r_{n+1} + 1 = m_n = 2^n\mu + 1$ and then $r_{n+1} = 2^n\mu$.

Theorem 3.2 *On an n-simplex, $r \ge 2^{n-1}\mu$ and $m \ge 2^n\mu + 1$. There exist schemes with $r = 2^{n-1}\mu$ and $m = 2^n\mu + 1$.*

4 Compatibility of Hermite interpolations with piecewise polynomials

4.1 THE ONE-DIMENSIONAL CASE

Now consider Hermite interpolation on splines. Suppose the interval $[a, b]$ is subdivided into n segments by inserting nodes $a = x_0 < x_1 < \cdots < x_q = b$. Define the spline space[7]

$$S^\mu_k(q) = \{s \in C^\mu : s|_{(x_{i-1}, x_i)} \in P_k, \ i = 1, 2, \cdots, q\}, \qquad (4.1)$$

which is linear with dimension

$$\dim S^\mu_k(q) = k + 1 + (q - 1)(k - \mu). \qquad (4.2)$$

Similar to the polynomial case, the Hermite interpolation for splines reads For $f \in C[a, b]$, find polynomial $s \in S^\mu_k(n)$ such that

$$\frac{d^t}{dx^t}s(x)=\frac{d^t}{dx^t}f(x),\ \text{for } x=a,b,\ \ 0\le t\le r. \quad (4.3)$$

Again, for a solution to this interpolation to exist, the number of conditions does not exceed the dimension of $S_k^\mu(q)$, i.e., $q(k-\mu)\ge 2r+1-\mu$.

Since a polynomial is a special case of a spline when its pieces become identical, we conclude that

Lemma 4.1 *The conditions in (4.3) are consistent if and only if*

$$q(k-\mu)\ge 2r+1-\mu\ . \quad (4.4)$$

If $q(k-\mu)=2r+1-\mu$, then there exists uniquely a spline $s\in S_k^\mu(q)$ satisfying (4.3).

4.2 THE TWO-DIMENSIONAL CASE

Suppose a triangle σ is subdivided into sub-triangles in such a way that each angle of σ is divided into parts and the edges of the sub-angles meet points inside σ that become interior vertices of the subdivision.

Now suppose the triangle σ is assigned data of order r at its vertices and data of order μ on its edges. Look at one vertex of σ, the corresponding angle of which is supposed to has been split into β parts. The layer $r+1$ of domain points at this vertex is related to a univariate Hermite interpolation in $S_{r+1}^\mu(\beta)$ with boundary data of order μ.

We continue to consider the compatibility of piecewise polynomial elements by using the ideas in the last subsection.

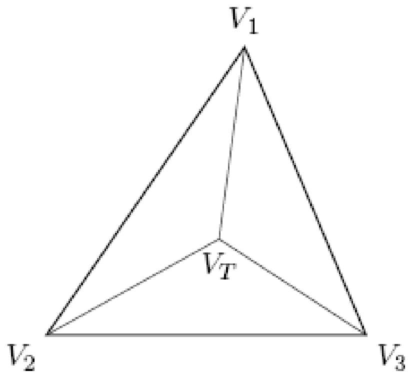

FIGURE 1 The Clough-Tocher split of a triangle.

Suppose an angle of the macro-triangle is split into β parts. The dada given at this vertex is of order r, while the global smoothness is μ. Look at the $r+1$ layer. Now it can be regarded as a univariate spline of degree $r+1$ and smoothness μ. The data on the two sides of the macro-triangle sharing this vertex induce that the univariate spline has data assigned of order μ at its two borders. In order for this Hermite interpolation to have a solution, it's necessary that $2(\mu+1)\le(r+2)+(\beta-1)(r+1-\mu)$.

Therefore we have the following result.

Theorem 4.2

$$\beta(r+1-\mu)\ge 1+\mu\ . \quad (4.5)$$

This inequality covers a variety of schemes in literature. Specifically,

(1) $\beta=1$. This implies $r\ge 2\mu$. This meets the case when the element is a polynomial and the macro-triangle is not split; see [3] and [4].

(2) $\beta=2$. This implies $2r\ge 3\mu-1$. The Clough-Tocher element and the Powell-Sabin element fall into this case; see [8], [9] and [10].

(3) $\beta=3$. This is the Morgan-Scott element [11].

(4) $\beta=4$. This is the Double-Clough-Tocher element; see [12]

(5) $\beta=\mu+1$ and $r=\mu$. This is the Shi-Wang element see [8].

5 Compatibility of Clough-Tocher elements

In this section, we will focus on the consistency analysis on the Clough-Tocher element of any dimensions. For this, we define $S_k^{\mu,r_0,\cdots,r_{n-2},\alpha}(\Delta_{CT})$ as the spline space on the Clough-Tocher macro-triangle Δ_{CT} with polynomial degree k, smoothness μ, data assigned on the i-faces of order r_i, $i=0,1,\ldots,n-2$, and super order α at the centroid. The objective is to setup the relation among these parameters for an interpolant to exist in this spline space.

5.1 TWO-DIMENSIONAL CASE

A Clough-Tocher split of a triangle is illustrated in Figure 1. Here $\beta=2$. From the above discussion in section 4.2, it holds $2r\ge 3\mu-1$, in which the smallest r can be taken as

$$r=\left\lceil\frac{3\mu-1}{2}\right\rceil=\begin{cases}3m, & \text{if } \mu=2m,\\ 3m+1, & \text{if } \mu=2m+1.\end{cases} \quad (5.1)$$

Let α be the super order enforced at the centroid. This is equivalent to that the local α influence region degenerates to a polynomial of degree α. Now face the edge A_1A_2 and increase α from 0 up to its ultimate value α such that the data it covers are compatible but it can't move further.

Notice that the local α influence region spans a triangle T_α by scaling the original triangle to α/q. The distance between a vertex of the local region is $\delta_v=q-\alpha$ and the order of data at the local vertex is $r-\delta_v$, while the order of data at the local edge is $\mu-\delta_v$. The compatibility may be destroyed at level $r-\delta_v+1$ of the local vertex. Therefore for the data to be consistent, we have

$$(r-\delta_v+1)+1\ge 2(\mu-\delta_v+1),$$

which, together with $q=2r+1$, reduces $3r+1-2\mu\ge\alpha$. To conclude, we have the following theorem.

Theorem 5.1 *On Clough-Tocher, there is a spline in $S_{2r+1}^{\mu,r,\alpha}(\Delta_{CT})$ with,*

$$r = \begin{cases} 3m, & \text{if } \mu = 2m, \\ 3m+1, & \text{if } \mu = 2m+1, \end{cases} \quad \alpha = \begin{cases} 5m+1, & \text{if } \mu = 2m, \\ 5m+2, & \text{if } \mu = 2m+1. \end{cases}$$

The figures from Figure 2 to Figure 6 illustrate the domain points of some splines on 2D Clough-Tocher split.

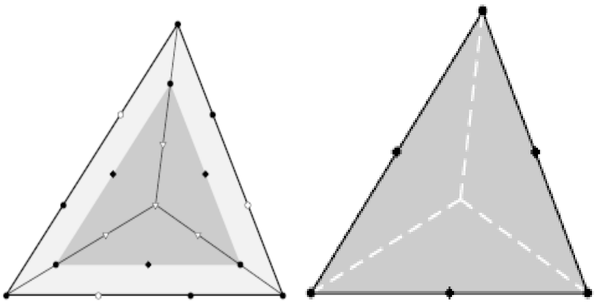

FIGURE 2. The domain points of $S_3^{1,1,2}(\Delta_{CT})$ with center remapped.

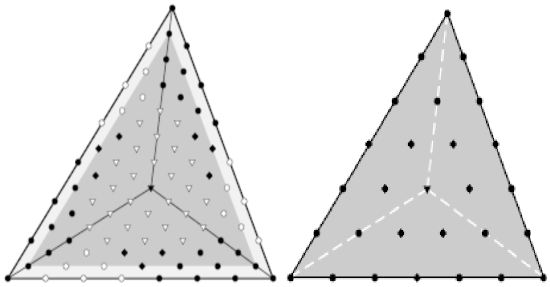

FIGURE 3. The domain points of $S_7^{2,3,6}(\Delta_{CT})$ with center remapped

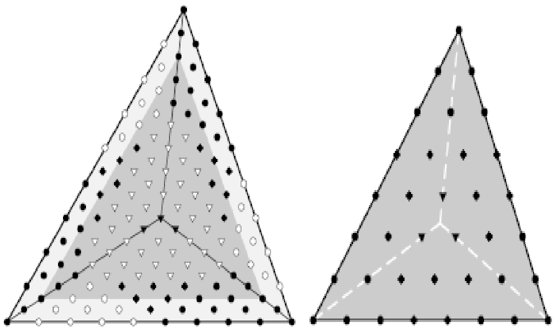

FIGURE 4. The domain points of $S_9^{3,4,7}(\Delta_{CT})$ with center remapped

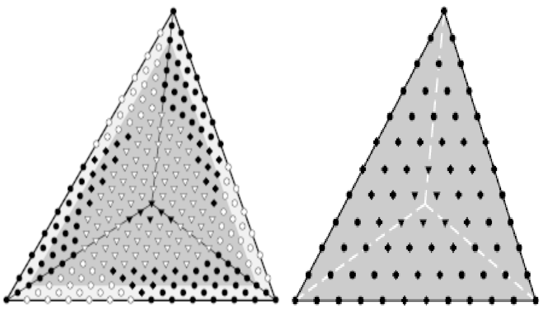

FIGURE 5. The domain points of $S_{13}^{4,6,11}(\Delta_{CT})$ with center remapped

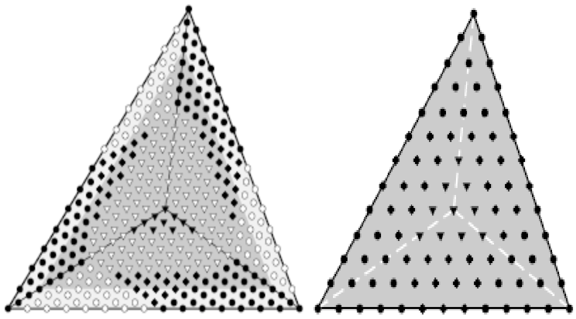

FIGURE 6. The domain points of $S_{15}^{5,7,12}(\Delta_{CT})$ with center remapped

5.2 THREE-DIMENSIONAL CASE

Now investigate the compatibility on the Clough-Tocher split of a tetrahedron $A_0A_1A_2A_3$. Suppose each piece of polynomials has degree p. Then $p \geq 2r_0 + 1$, where r_0 is the order of data assigned at vertices.

From the discussion in 2D case, look at the $r_0 + 1$-th layer to a vertex, which can be regarded as a 2D Clough-Tocher scheme with polynomial degree $r_0 + 1$, vertex data order r_1 and global smoothness μ . This gives

$$r_0 + 1 \geq 2r_1 + 1, \;\; 2r_1 \geq 3\mu - 1.$$

Further, to suppress freedom, we assume the interpolant has super smoothness $\alpha \geq \mu$ at the centroid. From this, we have

$$[r_1 - (n - \alpha) + 1] + 1 \geq 2\,[\mu - (n - \alpha) + 1],$$

$$[r_0 - (n - \alpha) + 1] + 1 \geq 2\,[r_1 - (n - \alpha) + 1].$$

which reduce to

$$r_1 + p - 2\mu \geq \alpha, \;\; r_0 + p - 2r_1 \geq \alpha .$$

Then based on (5.1) we can take,

$$p = 2r_0 + 1, \;\; r_0 = 2r_1, \;\; r_1 = \left\lceil \frac{3\mu - 1}{2} \right\rceil,$$

and

$$\alpha = r_1 + n - 2\mu = 5r_1 + 1 - 2\mu.$$

Therefore we have

Theorem 5.2 *On Clough-Tocher split of a tetrahedron, there is a spline in* $S_k^{\mu,r_0,r_1,\alpha}(\Delta_{CT})$, *where* $k = 2r_0 + 1$, $r_0 = 2r_1$,

$$r_1 = \begin{cases} 3m, & \text{if } \mu = 2m, \\ 3m+1, & \text{if } \mu = 2m+1. \end{cases}$$
$$\alpha = \begin{cases} 11m+1, & \text{if } \mu = 2m, \\ 11m+2, & \text{if } \mu = 2m+1. \end{cases} \tag{5.2}$$

5.3 GENERAL N-DIMENSIONAL CASE

From the above discussion, generally, a scheme of dimension n can be obtained via a scheme of dimension $n-1$. Because the methodology behind equation (5.2) still works for the general case, the following conclusion is obvious.

Theorem 5.3 *On Clough-Tocher split* Δ_{CT} *of a n-simplex with* $n > 3$, *there is a spline in* $S_k^{\mu,r_0,\cdots,r_{n-2},\alpha}(\Delta_{CT})$ *where* $k = 2r_0 + 1$, $r_i = 2r_{i+1},\ i = 0,\ 1,\ \cdots,\ r_{n-3}$,

$$r_{n-2} = \begin{cases} 3m, & \text{if } \mu = 2m, \\ 3m+1, & \text{if } \mu = 2m+1. \end{cases}$$

$$\alpha = \begin{cases} 11m+1, & \text{if } \mu = 2m, \\ 11m+2, & \text{if } \mu = 2m+1. \end{cases}$$

6 Conclusion

Interpolation over triangles is a fundamental technique in finite element approaches, where splines are preferable in serving as interpolants. Clough-Tocher macro-triangle is a typical split of a triangle due to its simple structure. We have discussed systematically in the above context on Clough-Tocher macro-elements of any smoothness in any dimension. A relation has been setup for smoothness, data orders and polynomial degrees. This relation is instructive in constructing an interpolation scheme over Clough-Tocher triangles. Further, we should investigate the performance of interpolants over Clough-Tocher triangle, and exploit interpolation over more sophisticated split structures.

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